Norm of bounded (continuous) linear transformations.

Definition: — Let N and N' be normed linear spaces and let T be a bounded linear transformation of N into N'. We define the norm of T by $\|T\| = \sup\{\|T(x)\| : x \in N, \|x\| \leq 1\}$. This norm is usually known as operator norm.

A Theorem: — Let N and N' be normed linear spaces and let T be a bounded linear transformation of N into N'. Put

$$a = \sup\{\|T(x)\| : x \in N, \|x\| = 1\}$$
$$b = \sup\{\|T(x)\| / \|x\| : x \in N, x \neq 0\}$$
$$c = \inf\{K : K \geq 0, \|T(x)\| \leq K\|x\| \ \forall x \in N\}.$$

Then $\|T\| = a = b = c$

and $\|T(x)\| \leq \|T\| \, \|x\| \ \forall x \in N$.

Proof: — By definition of norm

$$\|T\| = \sup\{\|T(x)\| : x \in N, \|x\| \leq 1\}.$$

By definition of c, $\|T(x)\| \leq c\|x\| \ \forall x \in N$

and if $\|x\| \leq 1$, then $\|T(x)\| \leq c \ \forall x \in N$.

And so $\sup\{\|T(x)\| : x \in N, \|x\| \leq 1\} \leq c$

i.e. $\|T\| \leq c$.

Also by definition of b and c it is clear that $c \leq b$.

Again if $x \neq 0$ then $\|T(x)\| / \|x\| = \|T(x / \|x\|)\|$ and $x/\|x\|$ has norm 1. Hence we conclude from the definitions of b and a that $b \leq a$. But it is evident that $a \leq \|T\|$. Thus we have show that

$$\|T\| \leq c \leq b \leq a \leq \|T\|$$

Which implies that $\|T\| = a = b = c$

Finally definition of b shows that

$$\|T(x)\| / \|x\| \leq b = \|T\|$$

i.e. $||T(x)|| \leq ||T|| \, ||x||$.

Theorem:- Let N and N' be normed linear spaces and let $B(N, N')$ denote the set of all bounded (or continuous) linear transformations from N into N'. Then $B(N, N')$ is itself a normed linear space with respect to pointwise linear operations.

$$(T+U)(x) = T(x) + U(x), \quad (\alpha T)x = \alpha T(x)$$

and the norm defined by

$$||T|| = \sup\{||T(x)|| : x \in N, ||x|| \leq 1\}.$$

Further if N' is a Banach space, then so is $B\{N, N'\}$.

Proof:- $B(N, N')$ is a linear space.

We know that the set S of all linear transformations from a linear space into another linear space is itself a linear space with respect to the pointwise linear operations. Therefore in order to prove that $B(N, N')$ is a linear space, it suffices to show that $B(N, N')$ is a subspace of S. Let $T_1, T_2 \in B(N, N')$. Then T_1, T_2 are bounded and so there exist real numbers $k_1 \geq 0$ and $k_2 \geq 0$ such that

$$||T_1(x)|| \leq k_1 ||x|| \text{ and } ||T_2(x)|| \leq k_2 ||x||$$

for all $x \in N$

If α, β are any two scalars, then

$$||(\alpha T_1 + \beta T_2)(x)|| = ||(\alpha T_1)(x) + (\beta T_2)(x)||$$

$= \| \alpha T_1(x) + \beta T_2(x) \|$

$\leq \| \alpha T_1(x) \| + \| \alpha T_2(x) \|$

$= |\alpha| \, \| T_1(x) \| + |\beta| \, \| T_2(x) \|$

$\leq |\alpha| k_1 \|x\| + |\beta| k_2 \|x\|$

$= (|\alpha| k_1 + |\beta| k_2) \|x\|$

Thus $\alpha T_1 + \beta T_2$ is bounded and so

$\alpha T_1 + \beta T_2 \in B(N, N')$

This proves that $B(N, N')$ is a linear subspace of S.

$B(N, N')$ is a normed linear space.

We verify the norm postulates one by one

(N1): Since $\|T\| = \operatorname{Sup}\{\|T(x)\| : \|x\| \leq 1\}$ and $\|T(x)\| \geq 0$.

We conclude that $\|T\| \geq 0$

(N2): We have $\|T\| = \operatorname{Sup}\{\|T(x)\| / \|x\| : x \in N, x \neq 0\}$.

$\therefore \|T\| = 0 \Rightarrow \operatorname{Sup}\{\|T(x)\| / \|x\| : x \in N, x \neq 0 = 0\}$.

$\Rightarrow \|T(x)\| / \|x\| = 0, x \in N, x \neq 0$

$\Rightarrow \|T(x)\| = 0, x \in N, x \neq 0$

$\Rightarrow T(x) = 0 \ \forall x \in N$

$\Rightarrow T = 0$ (zero transformation)

(N3): If $T, U \in B(N, N')$, then

$\|T+U\| = \operatorname{Sup}\{\|(T+U)(x)\| : x \in N, \|x\| \leq 1\}$.

$= \operatorname{Sup}\{\|T(x) + U(x)\| : x \in N, \|x\| \leq 1\}$.

$\leq \operatorname{Sup}\{\|T(x)\| + \|U(x)\| : x \in N, \|x\| \leq 1\}$.

$\leq \operatorname{Sup}\{\|T(x)\| : x \in N, \|x\| \leq 1\} + \operatorname{Sup}\{\|U(x)\| :$

$x \in N, \|x\| \leq 1\}$. $= \|T\| + \|U\|$.

(iii): If α is any scalar, then

$$\|\alpha T\| = \sup\{\|(\alpha T)(x)\| : x \in N, \|x\| \le 1\}$$
$$= \sup\{\|\alpha T(x)\| : x \in N, \|x\| \le 1\}$$
$$= \sup\{|\alpha|\,\|T(x)\| : x \in N, \|x\| \le 1\}$$
$$= |\alpha| \sup\{\|T(x)\| : x \in N, \|x\| \le 1\}$$
$$= |\alpha|\,\|T\|.$$

Hence $B(N,N')$ is a normed linear space.

$B(N,N')$ is complete if N' is complete.

Suppose N' is complete and let $\langle T_n \rangle_{n=1}^{\infty}$ be any Cauchy sequence in $B(N,N')$. Then

$$\|T_m - T_n\| \to 0 \text{ as } m,n \to \infty.$$

For each $x \in N$, we have

$$\|T_m(x) - T_n(x)\| - \|(T_m - T_n)(x)\|$$
$$\le \|T_m - T_n\|\,\|x\| \to 0$$

Hence $\langle T_n(x) \rangle$ is a Cauchy sequence in N' for each $x \in N$. Since N' is complete, there exists a vecter in N' which we denote by $T(x)$, such that $T_n(x) \to T(x)$. This defines a mapping T of N into N'. We now show that T is linear and bounded. If $x, y \in N$ and α, β are any scalars, then

$$T(\alpha x + \beta y) = \lim_{n \to \infty} T_n(\alpha x + \beta y)$$

$$= \lim_{n\to\infty} (\alpha T_n(x) + \beta T_n(y)) \quad [\because T_n \text{ is linear}]$$

$$= \alpha \lim_{n\to\infty} T_n(x) + \beta \lim_{n\to\infty} T_n(y) = \alpha T(x) + \beta T(y).$$

This shows that T is linear. To show that T is bounded, we observe that

$$\|T(x)\| = \|\lim T_n(x)\| = \lim \|T_n(x)\| \le \lim (\|T_n\| \, \|x\|)$$

$$\le \sup (\|T_n\| \, \|x\|)$$

$$= (\sup \|T_n\|) \, \|x\| \quad \text{---} (1)$$

Also we know that

$$| \|T_m\| - \|T_n\| | \le \|T_m - T_n\| \to 0 \text{ as } m, n \to \infty.$$

Therefore $\langle \|T_n\| \rangle$ is a Cauchy sequence of real numbers and hence convergent and bounded. So there exists $K \ge 0$ such that

$$\sup \|T_n\| \le K \quad \text{---} (2)$$

From (2) and (3), we have

$$\|T(x)\| \le K \|x\|$$

showing that T is bounded. In other words

$$T \in B(N, N').$$

Finally we show that $T_n \to T$. Let $\epsilon > 0$ be given. Since $\langle T_n \rangle$ is a Cauchy sequence there exists a positive integer m_0 such that

$$m, n \ge m_0 \Rightarrow \|T_m - T_n\| < \epsilon/2 \quad \text{---} (3)$$

Next, let $x \in N$ be such that $\|x\| \le 1$. Then since $T_n(x) \to T(x)$.

We can choose a positive integer $m_x \ge m_0$ such that

$$\|T(x) - T_{m_x}(x)\| \le \epsilon/2 \quad \text{---} (4)$$

Hence for all $n \geq m_0$ and $||x|| \leq 1$. We have

$$||T_n(x) - T(x)|| = ||T_n(x) - T_{m_x}(x) + T_{m_x}(x) - T(x)||$$

$$\leq ||T_n(x) - T_{m_x}(x)|| + ||T_{m_x}(x) - T(x)||$$

$$= ||(T_n - T_m)_x x|| + ||T_{m_x}(x) - T(x)||$$

$$\leq ||T_n - T_{m_x}|| \, ||x|| + ||T_{m_x}(x) - T(x)||$$

$$\leq ||T_n - T_{m_x}|| + ||T_{m_x}(x) - T(x)|| \quad ||x|| \leq 1]$$

$$< \epsilon/2 + \epsilon/2 \; || \; \epsilon \text{ by (3) and (4).}$$

Thus $||T_n(x) - T(x)|| < \epsilon \; \forall \; n \geq m_0$ and $x \in N$ such that $||x|| \leq 1$.

Hence $\sup\{||T_n(x) - T(x)|| : x \in N, ||x|| \leq 1\} < \epsilon \; (n \geq m_0)$

or $\sup(||T_n - T)x|| : x \in N, ||x|| \leq 1\} < \epsilon \; (n \geq m_0\}$.

or $||T_n - T|| < \epsilon \; (n \geq m_0)$

It follows that $T_n \to T$ and the proof is complete.

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